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Boolean Machinery for Quantum Logics

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The arithmetical tools based on Boolean matrices are described. They are applied to finite ortholattices to decompose them into products and sums, and to check atomisticity and orthomodularity.

1. INTRODUCTION

One can find finite ortholattices mentioned in most papers related to quantum logic. They are the basic source of examples (and counterexamples, too). Besides, if at some time quantum logic grows to be a calculational tool [as I hope happens sooner or later (Grib and Zapatrin, 1992; Zapatrin, 1993)], finite lattices could form the approximation medium for these future calculations. In any event, the machinery I describe was implicitly used in various lattice constructions, so its explicit exposure seems appropriate. This paper consists of the following.

The general polarity construction in the special case yielding finite ortholattices is described in Section 2.

Section 3 introduces the underlying Boolean matrices which are the basis of the proposed techniques.

Section 4 shows how the underlying matrices can be extracted from lattices described as orthoposets. The criterion is also established for an arbitrary Boolean matrix to be underlying for an ortholattice.

Section 5 is the first application of the proposed machinery: an algorithm is suggested which checks whether the lattice is atomistic.

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In Section 6 a more essential application is described: the recipe of decomposition of lattices into the cardinal products and horizontal sums of smaller lattices.

Section 7 provides the necessary and sufficient condition for an arbitrary finite ortholattice to be orthomodular. The proposed algorithm is exhaustive; however, it requires only one ranging over elements of the lattice in question.

2. POLAR LATTICES

In this section the polarity construction is described which is the standard source of complete orthocomplemented lattices.

Let V be a set equipped by a relation $\perp$ called *orthogonality* such that $\perp$ is *symmetric* ($u \perp v \;\Rightarrow\; v \perp u$) and *irreflexive* (there is no such $u \in V$ that $u \perp u$).

Definition 1.1. The *polar* to a subset $A \subseteq V$ is the subset $A^{\perp} \subseteq V$:

$$A^{\perp} := \{v \in V | \forall a \in A, a \perp v\} \tag{1.1}$$

Definition 1.2. The *closure* ClA of a subset $A \subset V$ is its bipolar:

$$ClA := A^{\perp\perp}$$

Definition 1.3. The collection $\Gamma = \Gamma(V)$ of all closed subsets of V is called a *polar lattice*.

Theorem 1.1. (i) The operation Cl is really closure, that is, for any $A, B \subseteq V$:

(C1) $A \subseteq ClA$.
(C2) $A \subseteq B$ implies $ClA \subseteq ClB$.
(C3) $ClClA = ClA$

(ii) $\Gamma(V)$ is the complete ortholattice: its partial order is set inclusion, and orthocomplements are polars (1.1).

Proof. See Birkhoff (1967, Chapter 5).

A subset V of a lattice L is called *join-dense* if any element $a \in L$ can be represented as a join of elements of V: $a = \bigvee \{v \in V | v \leq a\}$. Note that any lattice L contains join-dense subsets: as an example, it can be L itself (this will be used in Section 4).

Theorem 1.2. Let $(L, ')$ be a complete ortholattice, and V is a join-dense subset of L. Define the orthogonality relation $\perp$ on V:

$$u \perp v \quad \text{iff} \quad u \leq v'$$

Then the polar lattice $\Gamma(L)$ is orthoisomorphic to L.

Proof. See McLaren (1964).

Note that any lattice L contains join-dense subsets: L itself is an example of such a subset.

Definition 1.4. An element $a \in L$ is called an *amount* iff it is join-irreducible. Denote by V the set of all amounts of L:

$$a \in V \qquad \text{means} \quad a \neq \bigvee \{b \in L | b \leq a \ \& \ b \neq a\}$$

Any atom of L is amount (but not vice versa; for counterexample see the Example 4.1 below). When L is finite, the set V of all amounts is the least join-dense subset of L.

So, it follows from McLaren's (1964) theorem that any finite ortholattice L is unambiguously defined by the set V of its amounts and the orthogonality relation $\perp$ on V (see also Zapatrin, 1991, n.d.).

3. BOOLEAN MACHINERY

This section introduces two sorts of Boolean matrices, called underlying, associated with polarities and finite ortholattices. The criterion is established for an arbitrary Boolean matrix to be underlying for an ortholattice.

Let L be a finite ortholattice and V be the set of amounts of L. Two basic relations are defined on V. The first is the orthogonality $\perp$, and the second is its complement, the nonorthogonality relation [called sometimes accessibility, or possibility (Finkelstein and Finkelstein, 1983)]

$$uPv \qquad \text{iff} \quad \neg(u \perp v) \tag{3.1}$$

I introduce two kinds of matrices characterizing the lattice, called P-matrices and O-matrices:

Definition 3.1. The *underlying O-matrix* of a finite ortholattice L with the set of amounts V is the Boolean $|V| \times |V|$ matrix whose elements are defined as

$$\mathcal{O}_{uv} := \begin{cases} 1 & \text{if} \quad u \perp v \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.2. The *underlying P-matrix* for the ortholattice L is the Boolean $|V| \times |V|$ matrix associated with the relation P, (3.1):

$$\mathcal{P}_{uv} := \begin{cases} 1 & \text{if} \quad uPv \\ 0 & \text{otherwise} \end{cases}$$

To operate with Boolean matrices the Boolean arithmetics will be needed, possessing three operations:

1. Boolean sum: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$.
2. Boolean product: $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.
3. Negation: $\bar{0} = 1$, $\bar{1} = 0$ (or $\neg 0 = 1$, $\neg 1 = 0$).

These are distributive with respect to each other:

$$a(b + c) = ab + ac; \qquad a + bc = (a + b)(a + c)$$

and obey the de Morgan law:

$$\overline{a + b} = \bar{a} \cdot \bar{b}; \qquad (ab) = \bar{a} + \bar{b}$$

So the relationship between the underlying O- and P-matrices can be written as $\mathscr{P}_{uv} = \bar{\mathcal{O}}_{uv}$.

Now consider the requirements for an arbitrary square Boolean matrix to be the underlying P- or O-matrix for an ortholattice. The first necessary condition is that it must be symmetric and reflexive (to be a P-matrix), or irreflexive (to be an O-matrix). It turns out that the sufficient condition for P-matrices looks simpler, which is stated by the following theorem.

Theorem 3.1. A symmetric reflexive $|V| \times |V|$ Boolean matrix $\mathscr{P}$ is the underlying P-matrix for a finite ortholattice if and only if its rows (and hence columns) are Boolean linear independent. That is, there is no such row $\mathscr{P}_{u*}$ of $\mathscr{P}$ that it is the Boolean sum of some other rows different from $\mathscr{P}_{u*}$ (where $\mathscr{P}_{u*}$ means $\{\mathscr{P}_{uv} | v \in V\}$).

Proof. Since the amounts of L are join-irreducibles, the matrix $\mathscr{P}_{uv}$ is Boolean linear independent (to prove it, suppose the opposite; then the row which is the Boolean sum of some other rows is not an amount).

Now let $\mathscr{P}_{uv}$ be a reflexive symmetric matrix whose rows are Boolean linear independent. It suffices to prove that the amounts of the polar lattice generated by the polarity $(V, V, \perp)$, where $u \perp v$ means $\mathscr{P}_{uv} = 0$, are in one-to-one correspondence with the rows of the matrix $\mathscr{P}$. With any row P_{u*} the element of $\Gamma_\perp(V)$ is associated: $u \mapsto \{u\}^{\perp\perp}$. Hence, the set of amounts of Γ is contained among $\{u\}^{\perp\perp}$'s. Now suppose $\{u\}^{\perp\perp}$ is not an amount. This means that there is the collection of amounts $\{x\}^{\perp\perp}$, $\{y\}^{\perp\perp}, \ldots, \{z\}^{\perp\perp}$ such that $\{u\}^{\perp\perp} = \{x\}^{\perp\perp} \vee \{y\}^{\perp\perp} \vee \cdots \vee \{z\}^{\perp\perp}$. Since meets in Γ are set intersections, $\{u\}^\perp = \{x\}^\perp \cap \{y\}^\perp \cap \cdots \cap \{z\}^\perp$. Thus for any $v \in V$, $u \perp v$ if and only if for every $x, y, \ldots, z$, $x \perp v, y \perp v, \ldots,$ $z \perp v$. In terms of O-matrices one has

$$\forall v \quad \mathcal{O}_{uv} = \mathcal{O}_{xv} \mathcal{O}_{yv} \cdots \mathcal{O}_{zv}$$

Then the de Morgan rule yields

$$\forall v \quad \mathscr{P}_{uv} = \mathscr{P}_{xv} + \mathscr{P}_{yv} + \cdots + \mathscr{P}_{zv}$$

which means that the uth row of the matrix $\mathscr{P}$ is the Boolean sum of the rows $x, y, \ldots, z$.

Now let $(V, V, \perp)$ be an irreflexive symmetric polarity, and let $\mathscr{P}$ be the appropriate P-matrix (Definition 3.2).

Definition 3.3. A row $\mathscr{P}_{u*}$ of the $|V| \times |V|$ matrix $\mathscr{P}$ is called *redundant* if it is the Boolean sum of some other rows different from the uth. The element of the set V associated with this row is also called *redundant.*

Finally, Theorem 3.1 can be reformulated as follows: *Given a symmetric reflexive matrix $\mathscr{P}$, it is the P-matrix of an ortholattice if and only if it has no redundant rows.*

4. EXTRACTING OUT BOOLEAN MATRICES

In this section the algorithm is described based on redundancy criterion which extracts the underlying Boolean matrices from a lattice described as an orthoposet.

Now suppose that a finite ortholattice L is stored, say, in the memory of a computer as a partially ordered set. That is, all elements of L (without the least and the greatest one, since they are not proper elements) are enumerated by $1, \ldots, K$. The description of L as an orthoposet consists of two parts. The first is the partial order described as the matrix $\mathscr{G}$ of the relation $\geq$, namely

$$\mathscr{G}_{ik} = \begin{cases} 1 & \text{if} \quad i \geq k \\ 0 & \text{otherwise} \end{cases}$$

The second is the operation of orthocomplementation, which also can be defined as a matrix $\perp_{ik}$:

$$\perp_{ik} = \begin{cases} 1 & \text{if} \quad i = k^{\perp} \\ 0 & \text{otherwise} \end{cases}$$

Extracting Algorithm

1. Form the matrix product $\tilde{\mathscr{O}} := \perp \mathscr{G}$, one has

$$\tilde{\mathscr{O}}_{ik} = \sum_j \perp_{ij} \mathscr{G}_{jk}$$

2. Consider the matrix $\tilde{\mathscr{P}} = \neg \tilde{\mathscr{O}}$. These two steps can be contracted into one operation, namely

$$\tilde{\mathscr{P}}_{ik} = \prod_j \left(\bar{\perp}_{ij} + \bar{\mathscr{G}}_{jk} \right)$$

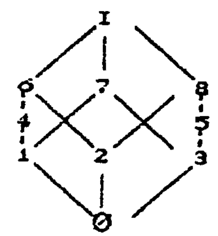
3. Using the redundancy theorem, delete all redundant rows and columns.

Theorem 4.1. The obtained reduced Boolean matrix, call it $\mathscr{P}$, will be just the P-matrix of the lattice L.

Proof. $\tilde{\mathscr{O}}_{ik} = 1$ means that $\exists j \; i^\perp = j \;\&\; j \geq k$, or $i \perp k$. Then, since L is join-generating for itself, perform the reducing procedure which will delete all nonamounts. So, the remainder will be just the underlying matrix for L.

Now it seems appropriate to introduce an elucidating example.

Example 4.1. Consider

$L =$ 

with orthocomplements defined as $1^\perp = 8$, $2^\perp = 7$, $3^\perp = 6$, $4^\perp = 5$. The matrices $\mathscr{G}$ and $\perp$ will have the form

$$\mathscr{G}_{ik} = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix} \qquad \perp_{ik} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Hence their product $\tilde{\mathscr{O}}$ and its complement $\tilde{\mathscr{P}}$ are

$$\tilde{\mathscr{O}}_{ik} = \begin{matrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \tilde{\mathscr{P}}_{ik} = \begin{matrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

The rows 6, 7, 8 are redundant, so the reduced matrix $\mathscr{P}$ has the form

$$\mathscr{P}_{ik} = \begin{matrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{matrix}$$

5. ATOMISTICITY TEST

In this section the Boolean machinery is applied to test whether a finite ortholattice L is atomistic.

Recall the necessary definitions. Atoms and amounts were already introduced in Section 2. A finite lattice L is called *atomistic* if all amounts of L are exhausted by atoms (Example 4.1 yields a nonatomistic lattice).

Atomisticity Test

1. Consider (or build, or extract) the P-matrix $\mathscr{P}$ of the lattice in question.

2. Check whether there exists a pair of rows i, k of $\mathscr{P}$ such that $\forall j, \mathscr{P}_{ij} \le \mathscr{P}_{kj}$, or in other terms, $\mathscr{P}_{i*} \le \mathscr{P}_{k*}$.

Criterion. If no pair of rows of $\mathscr{P}$ is comparable, then L is atomistic.

To prove that this criterion is right, note that the rows of $\mathscr{P}$ are in 1–1 correspondence with the amounts of L. If some pair of rows is comparable, that means that the greater one corresponds to a nonatomistic amount.

6. DECOMPOSITION OF LATTICES INTO SUMS AND PRODUCTS

In this section the cardinal products and horizontal sums of ortholattices are represented in terms of underlying matrices. The decomposition algorithm based on this representation is suggested. It also becomes clear why I had to introduce two sorts of underlying matrices.

Let L_1 and L_2 be two ortholattices with the greatest and the least elements I_1, I_2 and O_1, O_2, respectively.

Definition 6.1. The *cardinal product* (Birkhoff, 1967) $L_1 \times L_2$ is the collection of all pairs of elements from L_1 and L_2 with the partial order and orthocomplements defined element wise:

$$L_1 \times L_2 = \{(a_1, a_2) | a_1 \in L_1, a_2 \in L_2\}$$

$$(a_1, a_2) \le (b_1, b_2) \Leftrightarrow a_1 \le b_1 \ \& \ a_2 \le b_2$$

$$(a_1, a_2)^\perp = (a_1^\perp, a_2^\perp)$$

Definition 6.2. The *horizontal sum* $L_1 \oplus L_2$ is their disjoint set-theoretic union with the greatest and least elements pasted and the partial order and orthocomplements inherited from the summands:

$$L_1 \oplus L_2 = \{(a_1, 1), a_1 \in L_1 \backslash \{O_1, I_1\}\} \cup \{(a_2, 2), a_2 \in L \backslash \{O_2, I_2\}\} \cup \{O, I\}$$

$$(a_i, i) \le (b_i, i) \Leftrightarrow a_i \le b_i, \qquad i = 1, 2$$

$$(a_i, i)^\perp = (a_i^\perp, i), \qquad i = 1, 2$$

Now let V_1, V_2 be the sets of amounts of the lattices, and $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{O}_1, \mathcal{O}_2$ be their underlying matrices.

Theorem 6.1. Let $L = L_1 \times L_2$. Then the underlying P-matrix for L is the $(|V_1| + |V_2|) \times (|V_1| + |V_2|)$ matrix of the form $\mathcal{P} = \operatorname{diag}(\mathcal{P}_1, \mathcal{P}_2)$, or

$$\mathcal{P} = \begin{vmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{vmatrix} \tag{6.1}$$

Proof. First prove that the amounts of L are exhausted by the disjoint union $V_1 + V_2$. In fact, each amount (since it is join-irreducible) must contain 0 as one of the components. Hence, the other component is to be an amount of one of the lattices L_1 or L_2. Thus, the matrix $\mathcal{P}$ has the required cardinality. To prove that it has the required form, consider four possible cases:

1. $(u_1, O_2) \perp (v_1, O_2) \Leftrightarrow u_1 \perp v_1$.
2. $(O_1, u_2) \perp (O_1, v_2) \Leftrightarrow u_2 \perp v_2$.
3. $(v_1, O_2)^\perp = (v_1^\perp, I_2) \ge (O_1, v_2)$ for any $v_1 \in V_1$.
4. $(O_1, v_2)^\perp = (I_1^\perp, v_2) \ge (v_1, O_2)$ for any $v_2 \in V_2$.

Hence the O-matrix for L is

$$\mathcal{O} = \begin{vmatrix} \mathcal{O}_1 & 1 \\ 1 & \mathcal{O}_2 \end{vmatrix}$$

where 1 is the matrix having 1 in all entries. To complete the proof note that $\mathcal{P} = \bar{\mathcal{O}}$.

Theorem 6.2. Let $L = L_1 \oplus L_2$. Then the underlying O-matrix for L has the form $\mathcal{O} = \operatorname{diag}(\mathcal{O}_1, \mathcal{O}_2)$, or

$$\mathcal{O} = \begin{vmatrix} \mathcal{O}_1 & 0 \\ 0 & \mathcal{O}_2 \end{vmatrix} \tag{6.2}$$

Proof. In this case it is evident that the set of amounts of L is the same disjoint sum. To prove it has the required form, consider again four cases:

1, 2. $(v_i, i) \perp (u_i, i) \Leftrightarrow v_i \perp u_i$, $i = 1, 2$.
3, 4. No pair of elements from different components are orthogonal.

These two theorems form the basis for the decomposition algorithm. Its idea is the following. Considering both P- and O-matrices (now it is clear why I had to introduce two sorts of matrices) define whether the elements of V can be enumerated in such a way that $\mathscr{P}$ would have the form (6.1), or $\mathscr{O}$ would have the form (6.2). Then it can be claimed that the lattice L can be decomposed into a product (if the first possibility is realized) or a sum (if the second one holds), or it is not decomposable. The problem is to suggest the way to find out, given a Boolean matrix $\mathscr{R}$, whether it is decomposable. To do this, I suggest the following:

Matrix Decomposition Test. Let $\mathscr{R}$ be a symmetric reflexive $|V| \times |V|$ Boolean matrix. Consider the sequence $\mathscr{R}^1 = \mathscr{R}, \mathscr{R}^2 = \mathscr{R}\mathscr{R}, \mathscr{R}^3 = \mathscr{R}^2\mathscr{R}, \ldots$ (the matrix products). Since V is finite and $\mathscr{R}$ is reflective, starting from some $k \leq |V|$, $\mathscr{R}^{k+1} = \mathscr{R}^k$ (in terms of binary relations this means the transitive closure of the relation associated with $\mathscr{R}$). The obtained limit relation $\mathscr{R}^c$ is the equivalency on V. Then enumerate the elements of V beginning from the first equivalence class, exhausting the classes one after another. Then the matrix $\mathscr{R}'$ obtained from the initial $\mathscr{R}$ by appropriate permutation of rows and columns will have the form $\mathscr{R} = \mathrm{diag}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_K)$, where K is the number of equivalence classes.

Lattice Decomposition Algorithm

1. Suppose both P- and O-matrices for L are obtained. Then apply the matrix decomposition test to the matrices $\mathscr{P}$ and $(\mathscr{O} + \mathscr{E})$, where $\mathscr{E} = \mathrm{diag}(1, 1, \ldots, 1)$ is the unit $|V| \times |V|$ Boolean matrix. This decomposition test can be performed as calculating the $|V|$th powers of the matrices.

2a. If both $\mathscr{P}^c$ and $(\mathscr{O} + \mathscr{E})^c$ contain no zero entries, then L is NOT decomposable.

2b. If $\mathscr{P}$ is decomposable, $\mathscr{P} = \mathrm{diag}(\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_K)$, then $L = L_1 \times L_2 \times \cdots \times L_K$.

2c. If $\mathscr{O}$ is decomposable, $\mathscr{O} = \mathrm{diag}(\mathscr{O}_1, \mathscr{O}_2, \ldots, \mathscr{O}_K)$, then $L = L_1 \oplus L_2 \oplus \cdots \oplus L_K$.

3. To each of the components apply step 2.

Finally, the lattice will be decomposed into the sums and products of lattices of smaller cardinality.

7. ORTHOMODULARITY INQUIRIES

In this section the Boolean machinery provides the algorithm which tests the orthomodularity of the ortholattice.

First note that any exhaustive orthomodularity test needs ranging over elements of L, that is, over subsets of V. This is because the orthomodularity of a polar lattice is the second-order property of the orthogonality relation on the set V (Goldblatt, 1984). The theoretical ground for the proposed orthomodularity test is the result obtained by Foulis (1960). In the most suitable form in the sequel it is formulated as follows:

An ortholattice $(L, ^{\perp})$ is orthomodular if and only if for any $a \in L$ the mapping $\rho_a : L \mapsto L$ of the form

$$x\rho_a = (x \vee a^{\perp}) \wedge a \qquad \text{(the postfix notation is used)}$$

satisfies the condition

$$\forall u, v \in V \quad u\rho_a \leq v^{\perp} \Leftrightarrow v\rho_a \leq u^{\perp}$$

The idea of the proposed test is to express $u\rho_a \leq v^{\perp}$ as a binary relation, call it $\mathscr{D}_a$, on V, and then to prove that it is symmetric. Now let us build this relation using $\mathscr{P}, \mathscr{O}$, and the following operations with Boolean matrices:

1. Negation: $\bar{\mathscr{A}}_{ik} = \overline{(\mathscr{A}_{ik})}$.
2. Sum: $(\mathscr{A} + \mathscr{B})_{ik} = \mathscr{A}_{ik} + \mathscr{B}_{ik}$.
3. Matrix product: $(\mathscr{A}\mathscr{B})_{ik} = \sum \mathscr{A}_{ij}\mathscr{B}_{jk}$.
4. Pointwise product: $(\mathscr{A} \wedge \mathscr{B})_{ik} = \mathscr{A}_{ik}\mathscr{B}_{ik}$.

And one additional operation, denote it $\mathscr{A} \mapsto \mathscr{A}^o$, using the matrices $\mathscr{P}$ and $\mathscr{O}$:

5. $\mathscr{A}^o = \overline{\mathscr{A}\mathscr{P}} = \prod_j (\bar{\mathscr{A}}_{ij} + \mathscr{O}_{jk})$

Now, let a be an arbitrary element of L. Using the polar representation of L, one can consider a as the subset of V. Define the Boolean matrix $\mathscr{A}$ associated with a as follows:

$$\mathscr{A} = V \times \{v \in V | v \leq a\} \tag{7.1}$$

Lemma 7.1:

$$\mathscr{D}_a = (\mathscr{A} \wedge (\mathscr{O} \wedge \mathscr{A})^o)^o \tag{7.2}$$

Proof. The proof consists of stepwise development of the right side of the expression (7.2). The details of these techniques are in Zapatrin (1992).

To perform the orthomodularity test, the storage of Boolean vectors corresponding to elements of L (as polar lattice of subsets of V) must be prepared.

Orthomodularity Test

1. For $a \in L$ build the matrix $\mathscr{A} = V \times a$, (7.1).
2. Build the matrix $\mathscr{D}_a$, (7.2).
3. Check whether $\mathscr{D}_a$ is symmetric.

Criterion. L is orthomodular if and only if for every $a \in L$ the relation $\mathscr{D}_a$ is symmetric.

8. CONCLUDING REMARKS

The Boolean machinery is suggested as a working tool for finite ortholattices. As a matter of fact, the area of its application is broader than the proposed recipes. For example, if the underlying P-matrix $\mathscr{P}$ is considered not as a Boolean one but as the matrix over the trivial group $\{1\}$ with zero, then the Rees matrix semigroup having $\mathscr{P}$ as sandwich matrix will have the annihilator lattice isomorphic to the initial lattice L (Zapatrin, n.d.). In the case if L is orthomodular, it will be the Baer $*$-semigroup having L as lattice of its closed projectors (Foulis, 1960).

The extraction algorithm (Section 4) can be applied not only to lattices, but to arbitrary finite orthoposets. In this case the lattice restored from the obtained underlying matrix will be the McNeille completion of the initial poset.

Finally, I should mention that most of the results can be extended to infinite lattices (Zapatrin, 1992) and to lattices without orthocomplementation (Zapatrin, n.d.).

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